Diffusion on a curved surface:  
A geometrical approach

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Abstract

We propose a new model of 2D free particle diffusion on a possibly curved surface. This model is a generalization of the standard Ornstein-Uhlenbeck process and is completely determined by writing down the transport equation describing the diffusion in the phase-space of the diffusing particle. This transport equation is then used to show that curvature effects can profoundly affect the phenomenology of diffusion in the hydrodynamic limit. A specific pedagogical example is also worked out.

1 Introduction

Particle diffusion is probably the simplest of non-quantal irreversible phenomena. Surely the simplest model of particle diffusion in traditional infinite 3D Euclidean space is standard Brownian motion\([6]\). Let us suppose that the diffusion occurs within a fluid which is in a state of global thermodynamical equilibrium. It is then best to work in the Galilean reference frame where this fluid is at rest and, in this frame, the spatial density of a particle undergoing Brownian motion satisfies exactly the usual diffusion equation\([10]\). It is well-known that this model does not fix the time evolution of the diffusing particle momentum, since this quantity cannot even be defined for a particle undergoing Brownian motion (in the mathematical sense). This
represents a serious physical limitation which prompted Uhlenbeck and Ornstein to introduce, in 1930[12], the stochastic process which now carries their names. This process is usually defined by a couple of stochastic differential equations which fix the time-derivatives of both the position and momentum of the diffusing particle. The diffusion can then be described, at any time, by a probability distribution function in (one-particle) phase-space and the transport equation verified by this distribution function, the so-called Chapman-Kolmogorov (CK) equation, can be deduced without any approximation from the stochastic equations of motion[13, 10]. In the hydrodynamic (long-time) limit, the solutions to the Chapman-Kolmogorov equation have a spatial density which verifies approximately the usual diffusion equation[4]. Brownian motion in physical space thus appears as an approximate description of diffusion, valid for near-equilibrium situations only.

Let us now envisage a situation in which the 3D domain where diffusion occurs is not the whole Euclidean 3D space, but only a part of it. A general method for studying the behaviour of particles diffusing in this domain is obviously to use the standard diffusion equation or, if one wants to be more precise, the Chapman-Kolmogorov equation associated to the Ornstein-Uhlenbeck process, and to solve either of these equations with the appropriate boundary conditions.

Suppose now that the 3D domain under consideration is sufficiently thin to be validly approximated by a 2D surface. This situation commonly arises in surface sciences, where many phenomena involve particle diffusion in a thin interface which is often approximated by a surface. Similarly, biology offers many examples of diffusions in cellular membranes which may be considered as surface diffusions (although neglecting the membrane thickness can also lead to completely unrealistic models). In such cases, the 3D diffusion essentially comes back to a 2D diffusion on the surface approximating the thin 3D volume. How can one describe this 2D diffusion?

The general method outlined above can still be used; one then has to solve the 3D diffusion equation or the Kolmogorov equation with the proper boundary conditions associated to the thin 3D volume representing the membrane. But in the present case this very general method is obviously not optimal; indeed, solving the diffusion equation or the Chapman-Kolmogorov equation with arbitrary boundary conditions is certainly no easy task. Furthermore, the procedure would actually furnish more than what one is really interested in i.e., only the 2D aspects of the diffusion.

It is therefore best to start the analysis of the problem directly from a model of 2D diffusion. To the best of our knowledge, only one such model
has been proposed in the literature. This model is commonly known as “Brownian motion on a (Riemannian) manifold”[10] and is a direct generalization of the standard Brownian motion in 3D Euclidean space. As such, it suffers from the same drawbacks; in particular, Brownian motion on a surface only fixes the motion of the diffusing particle in physical space and furnishes no information about the motion in momentum- and phase-space because the momentum of a Brownian particle cannot even be defined.

To obtain a model of 2D diffusion which does not present the same limitations, it therefore seems natural to proceed exactly as Uhlenbeck and Ornstein did in the standard 3D case and to construct a 2D analogue of the Ornstein-Uhlenbeck process which would describe diffusion in a one-particle 4D phase-space. This process would naturally be associated to a generalized Chapman-Kolmogorov equation which would fix the time evolution of the probability density in this 4D phase-space. The aim of this article is to generalize the Ornstein-Uhlenbeck approach to 2D diffusion on an arbitrary surface and to discuss qualitatively how the curvature of the surface may affect the phenomenology of the diffusion.

This article is organized as follows. Section 2 reviews some basic facts about the usual Ornstein-Uhlenbeck process. Section 3 introduces the geometrical background necessary to the work that follows. Section 4 properly generalizes the standard CK equation to the case of diffusion occurring on an arbitrary curved 2D surface. Section 5 investigates some of the properties of this equation and of its solutions. In particular, it is shown that if the fluid which surrounds the diffusing particle is in a state of global thermodynamical equilibrium, the generalized CK equation admits an invariant measure susceptible of a natural physical interpretation; the first-order balance equations associated to the generalized CK equation are also derived. In Section 6, these balance equations are used to discuss qualitative aspects of diffusion phenomena on a curved surface. It is shown that various near-equilibrium regimes exist, depending on the order of magnitude of the curvature of the surface. For ‘small’ curvature, one recovers a regime whose properties are similar to those of standard Brownian motion. For ‘large’ curvature, the behaviors of the asymptotic regimes are markedly different from the behavior of Brownian motion. The differences are discussed at length in Section 6, where a simple but very instructive example is also fully worked out. Section 7 starts with a summary of the new results presented in this article and mentions some of their possible extensions.
2 The traditional Ornstein-Uhlenbeck process

2.1 General presentation

The well-known, standard Ornstein-Uhlenbeck process is a model of particle diffusion in 3D Euclidean space. Let \((x, p)\) be the position and momentum of the diffusing particle. For simplicity purposes, we will suppose that the fluid surrounding the diffusing particle is in a state of global thermodynamical equilibrium and we will work only in its rest (proper) frame. The Ornstein-Uhlenbeck process can be defined by the following couple of stochastic equations:

\[
\begin{align*}
\frac{dx}{dt} &= \frac{p}{m} \\
\frac{dp}{dt} &= -\alpha p dt + \sqrt{2D} dB_t
\end{align*}
\]

(1)

where \(m\) is the mass of the moving particle and \(\mu = \alpha m\) is the usual friction coefficient; \(B(t)\) stands for the standard 3D Brownian motion, the ‘derivative’ of which is a 3D Gaussian white noise. The amplitude of the noise is fixed by the positive coefficient \(D\), which is related to the standard diffusion coefficient \(D\) in physical space by \(D = \mu D\).

The one particle phase-space is \(\mathcal{P} = \mathbb{R}^6 = \{(x, p)\}\) and the volume measure on this phase-space is the Lebesgue measure \(d\Omega = d^3x d^3p\). Let \(\Pi(t, x, p)\) be the probability density in phase-space associated to equations (1). It obeys the Chapman-Kolmogorov equation:

\[
\partial_t \Pi + \nabla_x \cdot \left( \frac{p}{m} \Pi \right) + \nabla_p \cdot \left( -\alpha p \Pi \right) = D \Delta_p \Pi.
\]

(2)

The spatial density \(n(t, x)\), density current \(j(t, x)\) and stress-energy tensor \(T(t, x)\) of the diffusing particle are respectively defined by:

\[
n(t, x) = \int_{\mathbb{R}^3} \Pi(t, x, p) d^3p,
\]

(3)

\[
j(t, x) = \int_{\mathbb{R}^3} \frac{p}{m} \Pi(t, x, p) d^3p
\]

(4)

and

\[
T^{ik}(t, x) = \int_{\mathbb{R}^3} \frac{p^i p^k}{m} \Pi(t, x, p) d^3p.
\]

(5)
These tensor fields verify the following balance equations, which can be derived directly from (2):

$$\partial_t n + \partial_k j^k = 0 \quad (6)$$

and:

$$\partial_t j^k + \partial^l T^{kl} = -\alpha j^k. \quad (7)$$

Equation (6) expresses the so-called conservation of matter while (7) describes the net momentum exchange between the diffusing particle and the surrounding fluid.

The Ornstein-Uhlenbeck process admits as invariant measure, or stationary density, the Maxwellian:

$$\Pi_e(p) = (2\pi mk_B T_e)^{-3/2} \exp\left(-\frac{p^2}{2mk_B T_e}\right), \quad (8)$$

with:

$$T_e = \frac{D}{mk_B \alpha}. \quad (9)$$

Relation (9) is a fluctuation-dissipation theorem. Note that $\Pi_e$ is not normalized to unity in phase-space but in momentum space only.

### 2.2 Large-scale behavior of the Ornstein-Uhlenbeck process

From $m$, $T_e$ and $\alpha$, one can construct the following two quantities:

$$\tau = \frac{1}{\alpha} \quad (10)$$

and

$$\lambda = \tau \left(\frac{k_B T_e}{m}\right)^{1/2}. \quad (11)$$

$\tau$ is the characteristic time-scale of the process and $\lambda$ plays the role of a mean free-path.

The Chapman-Enskog method[2] can be used to study the process in the so-called hydrodynamic limit. More precisely, the Chapman-Enskog procedure delivers solutions to the CK equation which vary slowly in the scaled time- and space-variables $\theta = t/\tau$ and $\chi = x/\lambda$. One thus introduces[4] two infinitesimal parameters $\eta$ and $\epsilon$ such that:

$$\|\partial_t \Pi\| = \frac{1}{\tau} \|\Pi\|O(\eta) \quad (12)$$
and:
\[
\| \partial_\xi \Pi \| = \frac{1}{\lambda} \| \Pi \| O(\epsilon). \tag{13}
\]

The notation \( \| \cdot \| \) stands for an arbitrary norm defined over a function space to which \( \Pi \) belongs. One can use, for example:
\[
\| \Pi \| = \max_{(t, x, p) \in I \times D \times \mathbb{R}^3} \Pi(t, x, p), \tag{14}
\]

with \( I \) and \( D \) specifying the domain of space-time in which the solution has to be ‘slowly’ varying.

For physical reasons, one expects a slowly varying solution to be close to global equilibrium and, therefore, also close to local equilibrium. One thus introduces a third small parameter \( \epsilon' \) and writes:
\[
\Pi(t, x, p) = n(t, x) \Pi_0(p) + \sum_{q=1}^{\infty} \epsilon'^q \Pi_q(t, x, p). \tag{15}
\]

The balance equations (6) and (7) then imply the scaling relations \( \epsilon = \epsilon' \) and \( \eta = \epsilon^2 \). It is also found [4] that the perturbative solution (15) only exists if the density \( n \) verifies the usual diffusion equation:
\[
\partial_t n = \mathcal{D} \Delta n. \tag{16}
\]

As a conclusion to this Section, let us point out that the stochastic equations of motion (1), the phase-space \( \mathcal{P} \), the measure \( d\Omega \), the CK equation (2) and the diffusion equation (16) all are invariant under any change of coordinates which preserves the Euclidean metric, i.e. under an arbitrary isometry.

3 Geometrical considerations

The aim of this Section is certainly not to provide a primer in differential geometry, but rather to review some basic concepts which will be of constant use in the remainder of this article. These permit to obtain the desired results in a very general, formal way. The reader is referred to [11, 5, 16] for extensive pedagogical presentations of the material only broached upon in the present Section.
3.1 The spaces tangent and cotangent to a manifold

The natural mathematical concept to describe surfaces is that of differentiable manifold. Loosely speaking, a differentiable manifold of dimension \( n \) is a topological space which, at least locally, 'looks like' \( \mathbb{R}^n \). A surface is a two-dimensional differentiable manifold and, at least locally, one can use two real numbers as coordinates to label the points on the surface. These coordinates will be denoted by \( x^i, i = 1, 2 \). Occasionally, we will also write \( x = (x^1, x^2) \).

One will have to consider real-valued functions defined on the manifold. Given a coordinate system, any such function is described by a real-valued function \( f \) of the two coordinates \( x^1 \) and \( x^2 \). Let \( X \) be an arbitrary point on the surface. One can naturally evaluate the partial derivatives \( \partial_1 f \) and \( \partial_2 f \) at any point on the surface, and in particular at \( X \). But one can also evaluate the total derivative of \( f \) along an arbitrary curve \( x(s) \) which passes through \( X \). Let us choose the (otherwise arbitrary) parametrization in such a way that \( x(s = 0) = X \). Standard differential calculus then leads to:

\[
\frac{d}{ds} f(x^1(s), x^2(s))_{|s=0} = \frac{dx^1}{ds}_{|s=0} \partial_1 f_{|X} + \frac{dx^2}{ds}_{|s=0} \partial_2 f_{|X}. \tag{17}
\]

The quantities \( \frac{dx^1}{ds}_{|s=0} \) and \( \frac{dx^2}{ds}_{|s=0} \) correspond to what one would spontaneously call the 'components' of a vector tangent to the curve under consideration at \( X \). This remark can be built upon to define in quite general a manner what will be called a vector tangent to a manifold.

Let us fix again a point \( X \) on the manifold and consider the set of all sufficiently smooth curves passing through \( X \). By (17), each curve can be associated to a linear operator acting on the space of (real-valued) differentiable functions \( f \) defined over a neighborhood of \( X \). This operator is fully defined by the derivatives of \( x^1 \) and \( x^2 \) along the curve at point \( X \). The set of all these operators is called the space tangent to the manifold at point \( X \); it can be shown to be a real vector space. Moreover, this vector space is two-dimensional and admits the family (see (17)) \( (\partial_1, \partial_2) \) as a basis. This basis is called a coordinate basis. The components of an arbitrary tangent vector \( v \) at point \( X \) will be denoted by \( v^1 \) and \( v^2 \). Expression (17) thus represents the action of the vector \( v = (\frac{dx^i}{ds}_{|s=0}) \), tangent to the surface at point \( X \), on the function \( f \).

A tangent vector-field \( v(x) \) is simply a field of vectors, typically represented by two functions \( v^1(x^1, x^2) \) and \( v^2(x^1, x^2) \). Tangent vectors (or
vector-fields) are known as contravariant vectors (or contravariant vector-fields) in standard physics literature.

One will also have to use vectors cotangent to the manifold. The space cotangent to the manifold at point $X$ is the vector-space of real-valued one-forms defined over the tangent space at point $X$. In other words, a vector cotangent to the manifold at point $X$ is a linear application which transforms tangent vectors at point $X$ into real numbers. Such a form $w$ is fully determined by the images $w(\partial_1)$ and $w(\partial_2)$ of the tangent vectors $\partial_1$ and $\partial_2$, since these vectors form a basis of the tangent space at $X$. Let us introduce the two forms $I^1$ and $I^2$ defined by:

$$I^i(\partial_j) = \delta^i_j$$

(18)

for $(i, j) \in \{1, 2\}^2$. One can then write, for an arbitrary cotangent vector $w$ at $X$:

$$w = w(\partial_1)I^1 + w(\partial_2)I^2.$$  

(19)

This shows that $(I^1, I^2)$ constitutes a basis of the cotangent space at $X$; this basis is said to be dual to the coordinate basis $(\partial_1, \partial_2)$ used in the tangent space. This cotangent space is also two-dimensional, exactly as the tangent space. One commonly uses the notation $w_i = w(\partial_i)$ for the components of the cotangent vector $w$ in the basis $(I^1, I^2)$. In physics, a cotangent vector is traditionally called a covariant vector.

There is another important characterization of tangent and cotangent vectors and vector-fields. If one introduces new coordinates $x'^i$ on the manifold, the components $v'^i$ of a tangent vector-field $v$ in the new coordinate basis can be expressed in terms of the components $v^i$ of the same vector-field in the previous coordinate basis. Because of (17), one finds immediately:

$$v'^i = \frac{\partial x'^i}{\partial x^j} v^j.$$  

(20)

One also has, for an arbitrary cotangent vector-field $w$:

$$w'_{ij} = \frac{\partial x^j}{\partial x'^i} w_j.$$  

(21)

Tangent and cotangent vector-fields are examples of tensor-fields. A tensor-field of type $(0, 0)$ is simply a function defined on the manifold. It is sometimes called a scalar quantity. A tangent vector-field is a tensor-field of type $(1, 0)$ while a cotangent-vector is a tensor-field of type $(0, 1)$. A tensor-field $T$ of type $(2, 0)$ is represented, in each coordinate system, by
a set of functions $T^{ik}(x)$ which transform like the products $q^i q^k$ under an arbitrary change of coordinates. Similarly, a tensor-field $S$ of type $(0,2)$ is represented, in each coordinate system, by a set of functions $S_{ik}(x)$, which transform like the products $p_i p_k$ under an arbitrary change of coordinates. And a tensor $Q$ of type $(1,1)$ is characterized in a similar fashion by its mixed components $Q^i_k$. Similar definitions and coordinate-representations also apply to tensor-fields of other types.

Let us finally mention that the contraction operation can be used to construct new tensor-fields out of known ones and that this operation lowers the total order of tensor-fields. For example, the contraction:

$$v^k = \sum_{i=1}^{3} w_i T^{ik} = w_i T^{ik}$$

(22)

defines a tensor field $v$ of type $(1,0)$.

3.2 Metric and connection

3.2.1 Metric

A Riemannian metric $g$ is a tensor-field of order $(0,2)$ which is symmetric, positive definite and invertible. It can be represented, in any coordinate system, by its ‘covariant’ components $g_{ij} = g_{ji}$. Invertibility means that there exists a tensor-field $g^{-1}$, of order $(2,0)$, such that $g^{-1} g = g g^{-1} = 1$; in components, this reads:

$$(g^{-1})^{ij} g_{jk} = g_{kj}(g^{-1})^{ji} = \delta_k^i,$$

(23)

where $\delta_k^i$ stands for the usual Kronecker symbol. For simplicity reasons, one usually denotes the components $(g^{-1})^{ij}$ merely by $g^{ij}$. Note that (23) implies that the Kronecker symbol is a tensor of type $(1,1)$.

The metric can be used to establish a one-to-one correspondence between tangent and cotangent vector-fields. For example, if $v$ is a tangent vector-field, the components $\tilde{v}_i = g_{ij} v^j$ define a cotangent vector-fields.

Given a metric, it is therefore possible to define the (position-dependent) scalar product $u \cdot v$ of two tangent vector-fields $u$ and $v$ by the relation:

$$u \cdot v = \tilde{u}_i(x) v^i(x) = u^i(x) \tilde{v}_i(x) = g_{ij}(x) u^i(x) v^j(x).$$

(24)

The scalar product of two cotangent vector-fields can be defined in the same way. The metric $g(x)$ being positive definite at every point $x$, the scalar-product of a vector-field with itself is always non-negative and vanishes only at points where the vector-field itself vanishes.
Let \( v \) be again a vector-field tangent to the surface; suppose that \( v \) is normalized to unity. By definition, at any point \( X \), there exists a curve \( x(s) \) such that \( x(0) = X \) and \( \frac{dx^i}{ds} \big|_{s=0} = v^i(X) \). One then has:

\[
 g_{ij}(X)v^i(X)v^j(X) = g_{ij}(X) \frac{dx^i}{ds} \bigg|_{s=0} \frac{dx^j}{ds} \bigg|_{s=0} = 1. \tag{25}
\]

This shows that the parameter \( s \) retained along the curve under consideration verifies (in conventional notations):

\[
 ds^2 = g_{ij}dx^i dx^j. \tag{26}
\]

It is commonly said that (26) defines the proper-length at every point on the surface.

### 3.2.2 Connection

A connection is a tool which makes it possible to compare tangent (or cotangent) vectors defined at different points on the surface. This is for example necessary if one wants to define the derivative of a tangent (or cotangent) vector-field. Let us now concentrate on how this can be done.

It is straightforward to check that, even if \( v \) is a tangent vector-field, the quantities \( \partial_i v^j \) do not define a tensor. The remedy to this situation is to introduce a new field, called the connection and represented by the so-called connection coefficients \( \Gamma^k_{ij} \) and to define the covariant derivative \( \nabla v \) of a tangent vector-field \( v \) by the relation:

\[
 \nabla_i v^j = \partial_i v^j + \Gamma^j_{ik} v^k. \tag{27}
\]

A necessary and sufficient condition for \( \nabla v \) to be a tensor-field of type \( (1, 1) \) is that the connection coefficients obey the following transformation law under an arbitrary coordinate change \([8, 5]\):

\[
 \Gamma^k_{ij}' = \frac{\partial x^{k'}}{\partial x^k} \frac{\partial x^i}{\partial x'^j} \Gamma^k_{ij} + \frac{\partial^2 x^p}{\partial x'^i \partial x'^j} \frac{\partial x^{k'}}{\partial x^p}. \tag{28}
\]

The connection can also be used to define the covariant derivative of arbitrary tensor-fields; generally speaking, the covariant derivative of a tensor-field of type \( (r, s) \) is a tensor-field of type \( (r, s + 1) \). For the record, the covariant derivative of a cotangent vector-field \( w \) is given by \([5]\):

\[
 \nabla_i w_j = \partial_i w_j - \Gamma^k_{ij} w_k. \tag{29}
\]
The covariant derivatives of tensors of types \((2,0)\) and \((0,2)\) are defined by [8]:

\[
\nabla_i T^{jk} = \partial_i T^{jk} + \Gamma^j_{pi} T^{pk} + \Gamma^k_{pi} T^{pj} \tag{30}
\]

and

\[
\nabla_i T_{jk} = \partial_i T_{jk} - \Gamma^p_{ji} T_{pk} - \Gamma^p_{ki} T_{pj}. \tag{31}
\]

Note however that, if \(f\) is a scalar-field (function) defined over the surface, the quantities \((\partial_i f)\) do transform as the components of a cotangent vector-field. The covariant derivative of \(f\), also called the gradient of \(f\), thus coincides with the cotangent vector-field of components \((\partial_i f)\) in the basis \((I^1, I^2)\).

At this point of the presentation, metric and connection are two totally unrelated geometrical objects. However, each metric \(g\) can be used to define a particular connection canonically associated to it [5]. This connection is called the Levi-Civita connection (associated to the given metric) and, with it, the metric is covariantly constant: \(\nabla g = 0\). The coefficients of the Levi-Civita connection are given by:

\[
\Gamma^k_{ij} = \frac{1}{2} g^{kl} (\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij}). \tag{32}
\]

Note that these coefficients are symmetric in their lower indices.

The intrinsic curvature of a surface is characterized by a tensor \(R\) which depends on the connection; its components are defined by [5, 16]:

\[
R^l_{ijk} = \partial_j \Gamma^l_{ik} - \partial_i \Gamma^l_{jk} + \Gamma^p_{ik} \Gamma^l_{pj} - \Gamma^p_{jk} \Gamma^l_{pi}. \tag{33}
\]

The geometry of the standard 2D Euclidean space is characterized by the metric \(\eta\), the components of which are given, in orthonormal coordinates, by \((\eta_{ij}) = \text{diag}(1, 1)\). By (32), the connection coefficients associated to this metric thus vanish identically in these coordinates. Therefore, in orthonormal coordinates, covariant differentiation formally coincides with the usual partial differentiation. Note also that, because of the simplicity of the metric, \(\tilde{v}_i = v^i\) for \(i = 1, 2\). The distinction between tangent and cotangent vectors thus becomes somewhat academic. It is however necessary on non-flat surfaces (or on the flat 2D Euclidean space parametrized by non-orthonormal coordinates – for example, polar coordinates).

Since the coefficients of the Levi-Civita connection associated to the metric \(\eta\) vanish identically in orthonormal coordinates, so do the components of the curvature \(R\) in these coordinates. Since \(R\) is a tensor, this implies that its components vanish in all coordinate systems. The 2D Euclidean space is therefore flat.
4 Chapman-Kolmogorov equation on a surface

4.1 Choice of the phase-space

As can be gathered from the preceding considerations, the position of a particle diffusing on a surface will be represented by two coordinates \( x = (x^1, x^2) \) defined at least locally on the manifold which models the surface. The velocity \( \dot{x} \) of the diffusing particle is simply the tangent vector \( (dx^1/dt, dx^2/dt) \).

As for momentum-space at point \( x \), one can choose either the tangent space or the cotangent space at \( x \). Both choices lead naturally to equivalent physics. It is however technically preferable to use the cotangent space as momentum space. This choice is also more natural from a dynamical point of view. Indeed, momentum is best defined in analytical mechanics as the partial derivative of the Lagrangean with respect to the velocity. The Lagrangean \( L \) of a free particle of mass \( m \) moving on a surface is the scalar quantity:

\[
L(x, \dot{x}) = \frac{m}{2} g_{ij}(x) \dot{x}^i \dot{x}^j. \tag{34}
\]

The momentum \( p \), defined by:

\[
p_i = \frac{\partial L}{\partial \dot{x}^i} = m g_{ij} \dot{x}^j \tag{35}
\]

is thus a cotangent-vector. The Hamiltonian \( H \) corresponding to \( L \) is:

\[
H(x, p) = \frac{1}{2m} g^{ij}(x) p_i p_j; \tag{36}
\]

it is also a scalar.

For physical reasons, it seems best not to restrict the values of the two momentum components. The pair \((p_1, p_2)\) will thus describe \( \mathbb{R}^2 \).

Now to the volume-measure in phase-space. As in 3D Euclidean space, it will be the product of a measure \( d\Omega_x \) in physical space by a measure \( d\Omega_p \) in momentum-space. The natural measure \( d\Omega_x \) to be used on the surface itself is \([8, 11, 5, 7]\):

\[
d\Omega_x = \sqrt{\text{det} \ g} \ d^2 x, \tag{37}
\]

where \text{det} \( g \) stands for the determinant of the matrix made out of the components \( g_{ij} \) of \( g \). Thus defined, \( d\Omega_x \) is a scalar measure, independent of the coordinates chosen on the surface. Note that in 2D Euclidean space, \( \text{det} \ g = 1 \) in orthonormal coordinates.

The natural measure in momentum space is \([8, 7]\):

\[
d\Omega_p = \frac{1}{\sqrt{\text{det} \ g}} \ d^2 p. \tag{38}
\]
As $d\Omega_x$, $d\Omega_p$ is a scalar measure invariant under arbitrary coordinate changes. The measure on phase-space $\mathcal{P}$ is thus the scalar:

$$d\Omega = d\Omega_x \ d\Omega_p = d^2x \ d^2p.$$  

(39)

4.2 The probability distribution function in phase-space

Diffusion on a physical surface or 2D interface will be described by a transport equation verified by a time-dependent probability distribution function $\Pi$ defined on $\mathcal{P}$. Before writing down the transport equation, let us discuss some important properties of $\Pi$.

A basic (constitutive) assumption of the diffusion model constructed in this article is that its associated distribution $\Pi$ is a scalar quantity, invariant under arbitrary coordinate changes on the surface. The zeroth-moment of $\Pi$ is defined by:

$$n(t, x) = \int_{\mathbb{R}^2} \Pi(t, x, p) \ d\Omega_p$$  

(40)

and represents the spatial density of the diffusing particle. In particular, one has:

$$\int_{\mathcal{M}} n(t, x) \ d\Omega_x = 1$$  

(41)

at all times. In (41), $\mathcal{M}$ stands for the manifold which models the surface.

The diffusing particle current-density is defined by:

$$j_k(t, x) = \int_{\mathbb{R}^2} p_k \Pi(t, x, p) \ d\Omega_p;$$  

(42)

at any instant, $j$ is a field cotangent to the surface.

We will also need the stress-energy tensor $T$ of the diffusing particle, defined by:

$$T_{ik}(t, x) = \int_{\mathbb{R}^2} \frac{p_ip_k}{m} \Pi(t, x, p) \ d\Omega_p.$$  

(43)

4.3 Transport equation in phase-space

Let us now define a model of diffusion on 2D surfaces by writing down the transport equation verified by the associated probability distribution function in phase-space. This equation will be the simplest or minimal generalization of (2) to possibly curved 2D manifolds.

To make the discussion easier, it is convenient to rewrite the 2D version of equation (2) as $\mathcal{L}_e(\Pi) = 0$, where $\mathcal{L}_e$ is the differential operator defined
by:

\[ \mathcal{L}_e(\Pi) = \partial_t \Pi + \partial_i \left( \frac{\eta_{ij} p_j}{m} \Pi \right) + \partial_{p_i} (-\alpha p_i \Pi) - D \eta_{ij} \partial_{p_i} \partial_{p_j} \Pi \]  

(44)

and \( \eta_{ij} = \text{diag}(1,1) \) represents the components of the 2D Euclidean metric in orthonormal coordinates.

Physics has to be independent of the coordinate system chosen on the surface. This means that the transport equation has to be invariant under an arbitrary change of coordinates. We have imposed as a constitutive hypothesis of our model that \( \Pi \) itself is a scalar. Thus, in arbitrary coordinates, the equation should take the form \( \mathcal{L}(\Pi) = 0 \) where the new operator \( \mathcal{L} \) is a scalar operator which reduces to \( \mathcal{L}_e \) if one uses orthonormal coordinates in 2D Euclidean space.

The easiest way to obtain the simplest possible \( \mathcal{L} \) is to generalize each term in (44) separately.

The time-derivative \( \partial_t \) is trivially a scalar operator and can be conserved as such in \( \mathcal{L} \). As for the flat Euclidean metric \( \eta \), it naturally has to be replaced by the metric \( g \) of the manifold under consideration. The only remaining problem is to properly generalize the partial derivatives with respect to position and momentum.

To get a better understanding of the problem, let us consider an arbitrary field \( \phi \) defined over phase-space. To simplify the discussion further, we will suppose that \( \phi \) is a scalar field, invariant under arbitrary coordinate changes. Let us consider such a change of coordinates. Then, with obvious notations:

\[ \phi'(x', p') = \phi(x^i, p_j). \]  

(45)

Since:

\[ p_j = \frac{\partial x'^j}{\partial x^i} p_j', \]  

(46)

one has:

\[ \frac{\partial}{\partial x^i} \phi' = \frac{\partial x^i}{\partial x'^j} \frac{\partial \phi}{\partial x'^j} + \frac{\partial^2 x'^j}{\partial x^i \partial x^3} \frac{\partial x^i}{\partial x'^j} p'_j \frac{\partial \phi}{\partial p_j} \]  

(47)

and:

\[ \frac{\partial}{\partial p_j'} \phi' = \frac{\partial x'^j}{\partial x^i} \frac{\partial \phi}{\partial p_j}. \]  

(48)

Equation (48) proves that the operator \( \partial/\partial p_j \) transforms scalar fields into tangent vector fields. This can easily be generalized to fields of arbitrary tensor type. The notion of partial differentiation with respect to momentum
components is thus a perfectly generally covariant notion and the operator $\partial/\partial p_j$ transforms arbitrary tensor fields of type $\langle r, s \rangle$ into tensor fields of type $\langle r + 1, s \rangle$. This can be easily understood without any calculation if one notes that, at any point on the 2D surface, momentum space is a flat 2-dimensional manifold. This point deserves further comment.

Momentum space at $x$ is the space cotangent to the surface at $x$. This vector-space can be considered as a manifold in its own right. At an arbitrary `point’ $p$, this manifold has itself a tangent and a cotangent space. The natural ‘proper-length’ in momentum-space is defined by (see (26) :

$$d\sigma^2 = g^{ij}(x) \, dp_i dp_j; \quad (49)$$

there is indeed no other physically meaningful metric that can be used in momentum-space. This metric depends only on $x$ and not on $p$; the partial derivatives of $g^{ij}$ with respect to $p$ therefore vanish identically, and so do the coefficients of the Levi-Civita connection associated to this metric in momentum space. Covariant differentiation with respect to momentum-components at fixed $x$ therefore comes back to partial differentiation with respect to the same components.

On the other hand, equation (47) proves that, even for scalar fields, partial differentiation with respect to one of the spatial coordinates in not a covariant notion. This again is quite simple to understand.

In passing from $x$, to $x+dx$, the usual partial differentiation with respect to the coordinates $x^i, i = 1, 2$, maintains the components of $p$ constant. In flat Euclidean space and orthonormal coordinates, this amounts to maintaining the vector $p$ itself constant. But on an arbitrary surface, the basis $(I^1, I^2)$ in momentum-space is itself position-dependent. This can be seen in the following manner. Let $T$ be an arbitrary tensor-field. The only coherent notion of differentiation that exists on an arbitrary surface is covariant differentiation. In particular, partial differentiation of a tensor-field is not even defined in differential geometry\footnote{One could think of defining the partial derivative of a tensor-field as a new field which would be represented, in a given coordinate-system, by the partial derivatives of the components of $T$. The problem is that this new field would not be a tensor-field. As a consequence, the new field could for example vanish identically in a given coordinate system and not in another one.} Therefore, an arbitrary tensor-field is position-independent (in any coordinate system) if it is covariantly constant i.e. if its covariant derivatives with respect to all spatial coordinates vanish identically. This is the only definition that makes sense independently of the chosen coordinate system.
Let us come back to the basis cotangent vectors $I^1$ and $I^2$. By definition, their components are respectively $(1, 0)$ and $(0, 1)$. By equation (29), the components of $\nabla I^1$ and $\nabla I^2$ are given by:

\[
(\nabla I^1)_{ij} = \nabla_i (I^1)_j = -\Gamma^1_{ij} \tag{50}
\]

and

\[
(\nabla I^2)_{ij} = \nabla_i (I^2)_j = -\Gamma^2_{ij}. \tag{51}
\]

The basis in momentum space is therefore position dependent and maintaining a momentum constant is not equivalent to maintaining its components constant.

‘Differentiation with respect to position at constant momentum’ has therefore to be defined as covariant differentiation with respect to position (i.e. $\partial \rightarrow \nabla$) at momentum covariantly constant[7]. Equation (29) shows that the variations of the components of a covariantly constant cotangent vector-field $\rho$ between points $x$ and $x + dx$ are given by:

\[
d\rho_j = \Gamma^k_{ij} \rho_k dx^i \tag{52}
\]

One is therefore led to define a new operator $D_i$, acting on cotangent vector-fields defined over the 4D phase-space, by:

\[
D_i = \nabla_i + \Gamma^k_{ij} p_k \frac{\partial}{\partial p^j}, \tag{53}
\]

where $\nabla$ stands for the usual covariant derivative operator with respect to spatial degrees of freedom. This operator, which behaves as a covector under an arbitrary change of coordinates, transforms tensor-fields of type $(r, s)$ into tensor-fields of type $(r, s + 1)$. Using $D_i$, one can then define the transport operator $L$ by:

\[
L(\Pi) = \partial_t \Pi + D_i (g^{ij}(x) \frac{p_j}{m} \Pi) + \frac{\partial}{\partial p_i} (-\alpha p_i \Pi) - Dg_{ij}(x) \partial_{p_j} \partial_{p_i} \Pi \quad \tag{54}
\]

and the transport equation verified by $\Pi$ is simply $L(\Pi) = 0$. Its most useful form is:

\[
\partial_t \Pi + D_i (g^{ij}(x) \frac{p_j}{m} \Pi) = \frac{\partial}{\partial p_i} \left( \alpha p_i \Pi + Dg_{ij}(x) \partial_{p_j} \Pi \right) \tag{55}
\]

This equation describes the statistics of a point-like particle moving on a 2D surface and diffusing because of its interaction with a surrounding fluid. This fluid has been implicitly supposed to be in a state of global thermodynamical equilibrium characterized by a velocity $V = 0$ in the chosen reference frame and by a temperature $T_e$ whose expression in terms of $m, \alpha$ and $D$ will be determined in the next section.
5 Basic properties of the model

5.1 Invariant measure

For physical reasons, one expects the CK equation to admit a time-independent solution; this equilibrium solution should be a generalization of the Maxwellian distribution given by (8). A natural candidate for a time-independent equilibrium solution to the CK equation is therefore:

$$\Pi_e(x, p) = (2\pi mk_BT_e)^{-1} \exp\left(-\frac{H(x, p)}{k_BT_e}\right),$$  \hspace{1cm} (56)

where $H(x, p)$ has been defined in (36). $H$ is the scalar position- and momentum-dependent Hamiltonian of a particle moving freely on the surface and $T_e$ is a characteristic temperature whose dependence on the three parameters $m$, $\alpha$ and $D$ is left open at this stage. Physically, $T_e$ represents the temperature of the fluid in which the particle diffuses. The factor in front of the exponential in (56) ensures that $\Pi_e$ is normalized to unity in momentum-space.

Let us now check that $\Pi_e$ is indeed a solution of the CK equation (55). One sees immediately that $\Pi_e$ defined by (56) makes the right-hand side of equation (55) vanish if the temperature $T_e$ is linked to $\alpha$ and $D$ by the fluctuation-dissipation relation:

$$T_e = \frac{D}{mkB\alpha},$$  \hspace{1cm} (57)

which is identical to (9).

Let us now verify that the left-hand side of (55) also vanishes for $\Pi = \Pi_e$. One has:

$$D_k(g^{ij}(x)) = \nabla_k(g^{ij}(x)) = 0,$$  \hspace{1cm} (58)

because the connection is the Levi-Civita connection associated to the metric $g$. Hence:

$$D_i\left(g^{ij}(x)p_j\Pi_e\right) = g^{ij}(x)\left(D_i(p_j)\exp\left(-\frac{H(x, p)}{k_BT_e}\right) + p_jD_i\left(\exp\left(-\frac{H(x, p)}{k_BT_e}\right)\right)\right).$$  \hspace{1cm} (59)

Since the $x^i$’s and the $p_j$’s are independent variables, one naturally has $\partial_ip_j = 0$ for any value of $(i, j)$; equation (29) then leads to:

$$\nabla_ip_j = \partial_ip_j - \Gamma^k_{ij}p_k = -\Gamma^k_{ij}p_k.$$  \hspace{1cm} (60)
On the other hand,
\[ \Gamma^l_{ik} p_l \partial_{p_k} p_j = \Gamma^l_{ik} \Gamma^l_{pj} = \Gamma^l_{ij} p_l. \] (61)
Combining (61) and (61), one finds that \( D_i(p_j) = 0. \)

The only possibly non-vanishing contribution to (59) is therefore the one that involves the generalized derivative \( D_i \) of the exponential. Replacing \( D_i \) by its expression (53), one is led to evaluate two apparently very different terms. The first one is the covariant derivative \( \nabla_i \) of the exponential function. This function is a scalar, like \( H \). Its covariant derivatives therefore coincide with its partial derivatives. One thus has:

\[ \nabla_i \exp \left( - \frac{H(x, p)}{k_B T_e} \right) = - \frac{p_j p_k}{2 m k_B T_e} \partial_j g^{jk} \exp \left( - \frac{H(x, p)}{k_B T_e} \right). \] (62)

The other contribution to the generalized derivative \( D_i \) of the exponential involves partial derivatives with respect to the momentum components. It reads:

\[ \Gamma^l_{ik} p_l \partial_{p_k} \exp \left( - \frac{H(x, p)}{k_B T_e} \right) = - \frac{m}{k_B T_e} p_j g^{jk}(x) \Gamma^l_{ik} p_l \exp \left( - \frac{H(x, p)}{k_B T_e} \right). \] (63)

The relation \( \nabla g = 0 \) translates into:

\[ \partial_i g^{jk} = - \Gamma^l_{mj} g^{mk} - \Gamma^l_{m} g^{jm}. \] (64)

If one uses (64) to eliminate the partial derivatives of the metric components from (62) and then combines the result with (63), one obtains immediately:

\[ D_i \left( \exp \left( - \frac{H(x, p)}{k_B T_e} \right) \right) = \left( \nabla_i + \Gamma^l_{ik} p_l \partial_{p_k} \right) \left( \exp \left( - \frac{H(x, p)}{k_B T_e} \right) \right) \]
\[ = 0. \] (65)

This completes the proof that \( \Pi_e \) is indeed a solution of the CK equation introduced in the preceding section.

5.2 Balance equations

5.2.1 Matter conservation

Direct integration of the CK equation (55) against the measure \( d\Omega_p \) leads to:

\[ \partial_t n + \int_{\mathbb{R}^3} D_i \left( g^{ij}(x) \frac{p_j}{m} \Pi \right) d\Omega_p = 0. \] (66)
Because of the very definition of $D_i$:

$$\int_{\mathbb{R}^2} D_i \left( g^{ij}(x) \frac{p^j}{m} \Pi \right) d\Omega_p = \int_{\mathbb{R}^2} \left[ \partial_i (p^j \Pi) + \Gamma^i_{jk} p^k \Pi + \Gamma^i_{jk} \partial_j (p^j \Pi) \right] d\Omega_p,$$

where the short-hand notation $p^j = g^{ij}(x)p_j$ has been used. Contrary to what happens in the usual Euclidean case, the partial derivative $\partial_i$ appearing in the first contribution to the right-hand side of (67) cannot be taken directly out of the integral because the measure $d\Omega_p$ is itself position-dependent. Indeed:

$$\partial_i (d\Omega_p) = \partial_i \left( \frac{1}{\sqrt{\det g}} \right) d^2 p = -\Gamma^k_{ik} d\Omega_p.$$  

(68)

This leads to:

$$\int_{\mathbb{R}^2} \partial_i (p^j \Pi) d\Omega_p = \partial_i \int_{\mathbb{R}^2} p^j \Pi d\Omega_p + \int_{\mathbb{R}^2} \Gamma^k_{ik} p^j \Pi d\Omega_p.$$  

(69)

To proceed further, let us integrate by part the last contribution to the right-hand side of (67). One finds:

$$\int_{\mathbb{R}^2} \Gamma^k_{ij} p^j \partial_i (p^j \Pi) d\Omega_p = -\int_{\mathbb{R}^2} \Gamma^k_{ik} p^j \Pi d\Omega_p.$$  

(70)

Combining equations (67), (69) and (70), one obtains:

$$\int_{\mathbb{R}^2} D_i \left( g^{ij}(x) \frac{p^j}{m} \Pi \right) d\Omega_p = \partial_i \int_{\mathbb{R}^2} p^j \Pi d\Omega_p + \int_{\mathbb{R}^2} \Gamma^i_{jk} p^j \Pi d\Omega_p.$$  

(71)

Inserting the result into (66) leads to:

$$\partial_t n + \nabla_i \left( g^{jk} j_k \right) = 0,$$

(72)

which expresses matter-conservation for the diffusing particle.

### 5.2.2 Momentum exchanges with the fluid

By a similar but rather more tedious calculation, the first moments of the CK equation lead to:

$$\partial_t j_k + \nabla_i \left( g^{ij} j_k \right) = -\alpha j_k.$$  

(73)

This equation describes the balance of momentum exchanges between the diffusing particle and the fluid by which it is surrounded.
6 Qualitative analysis of diffusion on a curved surface in the hydrodynamic limit

6.1 General Discussion

The aim of this section is to discuss qualitatively the behaviour of near-equilibrium solutions to the CK equation on a curved surface. As in the standard Euclidean flat case, we will write:

\[ \Pi(t, x, p) = \Pi_0(t, x, p) + \Delta \Pi(t, x, p), \]  

(74)

with:

\[ \Pi_0(t, x, p) = n(t, x) \Pi_e(x, p) \]  

(75)

\[ \Delta \Pi = O(\epsilon') \Pi_e \]  

(76)

and

\[ \int_{\mathbb{R}^2} \Delta \Pi(t, x, p) d\Omega_p = 0 \]  

(77)

for all \( t \) and \( x \). The infinitesimal \( \epsilon' \) thus measures the departure from local equilibrium.

The spatial density associated to \( \Pi_0 \) is naturally \( n_0(t, x) = n(t, x) \), the current \( j_0 \) vanishes identically and the stress-energy tensor \( T_0 \) reads:

\[ (T_0)_{ik}(t, x) = n(t, x) \frac{k_B T_e}{3} g_{ik}(x) = n(t, x) \frac{\lambda^2}{3\tau^2} g_{ik}(x), \]  

(78)

where the characteristic time \( \tau \) and the mean free-path \( \lambda \) are defined by the same equations (10) and (11) as in the flat Euclidean case.

On physical grounds, one also expects that, for each of these solutions, there is a time-interval \( I \) and a domain \( D \) on the surface over which \( \Pi \) is a ‘slowly’ varying function of \( t \) and \( x \). As in Euclidean space, slow temporal variation is enforced by the condition:

\[ \| \partial_t \Pi \| = \frac{1}{\tau} \| \Pi \| O(\eta), \]  

(79)

where \( \eta \) is another infinitesimal. The norm \( \| \cdot \| \) can for example be defined by:

\[ \| \Pi \| = \max_{(t, x, p) \in I \times D \times \mathbb{R}^2} \Pi(t, x, p), \]  

(80)

Equation (79) implies:

\[ \| \partial_t n \| = \frac{1}{\tau} \| n \| O(\eta). \]  

(81)
Because of the curvature effects, one has to be cautious in defining what ‘slow spatial variation’ actually means. Since the scaling laws obeyed by diffusion in the hydrodynamic limit can be deduced from the balance equations introduced in Section 5, let us concentrate our analysis on the various contributions to these equations.

Let us start with the two terms contributing to the continuity equation (72). The density \( n \) is a scalar-field and its covariant derivatives coincide with its partial derivatives. It is therefore natural to enforce slow spatial variation of \( n \) by the same relations as in Euclidean space:

\[
\| \partial_x n \| = \frac{1}{\lambda} \| n \| O(\epsilon), \tag{82}
\]

\( \epsilon \) being yet another infinitesimal. Condition (82) is invariant under coordinate changes for which \( |dx'/dx| = O(1) \).

Now to the covariant divergence of \( j \). For a vector-field, the covariant derivatives do not coincide with the partial derivatives. One therefore has to introduce another quantity \( \nu \) which fixes the scale of the covariant derivative acting on \( j \). As already noted, the current \( j_0 \) associated to \( \Pi_0 \) vanishes identically; the total current \( j \) is thus of order \( \epsilon' \). More precisely:

\[
\max_k \| j_k \| = \frac{\lambda}{\tau} O(\epsilon'). \tag{83}
\]

One therefore has:

\[
\| \nabla_i (g^{ik} j_k) \| = \frac{1}{\tau} O(\nu \epsilon') \tag{84}
\]

As (82), (84) is invariant under coordinate changes for which \( |dx'/dx| = O(1) \). If the surface is flat, \( \nu = \epsilon \); but both quantities are generically independent, because \( \nabla \) takes into account curvature effects through the coefficients \( \Gamma \) of the connection. One can thus have \( \nu < \epsilon \) as well as \( \nu > \epsilon \).

The continuity equation (72), combined with (81) and (84), leads to the following scaling law:

\[
\eta = \nu \epsilon'. \tag{85}
\]

Let us now work on the momentum-balance. By (83) and (79), the scaling for the time-derivative of \( j \) is:

\[
\max_k \| \partial_t j_k \| = \frac{\lambda}{\tau^2} O(\eta \epsilon'). \tag{86}
\]

The total stress-energy tensor can be written as:

\[
T_{ik} = (T_0)_{ik} + \Delta T_{ik}, \tag{87}
\]
with \( T_0 \) defined by \((78)\) and \( \Delta T = T_0 \mathcal{O}(\epsilon') \). One has therefore:

\[
\nabla_i \left( g^{ij} T_{jk} \right) = \nabla_i \left( g^{ij}(T_0)_{jk} \right) + \nabla_i \left( g^{ij} \Delta T_{jk} \right).
\]

(88)

Since the covariant derivatives of the metric-tensor vanish, \((78)\) implies:

\[
\nabla_i \left( g^{ij}(T_0)_{jk} \right) = \left( \frac{\partial_i n}{n} \right) g^{ij}(T_0)_{jk} = \frac{\lambda}{\tau^2} \mathcal{O}(\epsilon).
\]

(89)

Generically:

\[
\nabla_i \left( g^{ij} \Delta T_{jk} \right) = \frac{\lambda}{\tau^2} \mathcal{O}(\nu \epsilon').
\]

(90)

The momentum-balance equation thus contains terms of order \( \eta \epsilon', \epsilon, \nu \epsilon' \) and \( \epsilon' \). This implies that:

\[
\epsilon = \epsilon'.
\]

(91)

Equations \((85)\) and \((91)\) are the two scaling laws which characterize the hydrodynamic limit of particle diffusion on a surface. The crucial difference with diffusion in flat Euclidean space lies in the \( \nu \) parameter. This parameter takes into account the spatial inhomogeneity of \( n \) and the curvature effects which stem from the spatial inhomogeneity of the metric. The temporal variation of the density is therefore fixed by the spatial inhomogeneities of both the density and the metric.

There are obviously two extreme regimes. If curvature effects are negligible, the parameter \( \nu \) coincides with \( \epsilon, \eta \) therefore coincides with \( \epsilon^2 \) and, at least at leading order, the diffusion behaves as diffusion in flat space. At the other end of the spectrum, one can envisage situations where the curvature effects, which code for the spatial inhomogeneity of the metric, dominate over the spatial inhomogeneity of the density; those cases correspond to \( \nu \gg \epsilon \) and, therefore, \( \eta \gg \epsilon \). For example, if \( \nu = \mathcal{O}(1) \), \( \eta = \epsilon = \epsilon' \). This makes clear that diffusion on a surface can be phenomenologically very different from diffusion in flat Euclidean space.

### 6.2 Example

Let us now illustrate these general theoretical results on as simple an example as possible.

Let \((O, x, y, z)\) be orthonormal coordinates in 3D Euclidean space. Consider the surface \( \Sigma \) defined by:

\[
x = R(z) \cos \theta \\
y = R(z) \sin \theta,
\]

(92)
where \( R(z) \) is a positive definite function of \( z \in I \), \( I \) an arbitrary interval in \( \mathbb{R} \) and \( \theta \in [0, 2\pi[. \) \( \Sigma \) is a cylinder of altitude-dependent radius. We will illustrate the discussion of the preceding Section by envisaging a simple case of particle diffusion on \( \Sigma \).

A convenient coordinate system on \( \Sigma \) is \((z, \theta)\). In these coordinates, the line element on \( \Sigma \) takes the form:

\[
d s^2 = \left( 1 + \left( R'(z) \right)^2 \right) dz^2 + R^2(z) d\theta^2
\]

(93)

The metric (93) is everywhere regular and invertible. The inverse metric is given by:

\[
g^{ij} = \text{diag} \left( \left( 1 + \left( R'(z) \right)^2 \right)^{-1}, R^{-2}(z) \right).
\]

(94)

A direct calculation shows that, in these coordinates, the only non-vanishing connection coefficients are:

\[
\Gamma^\theta_{\theta z} = \Gamma^\theta_{z \theta} = \frac{R'}{R},
\]

(95)

\[
\Gamma^z_{\theta \theta} = -\frac{RR'}{1+R'^2}
\]

(96)

and:

\[
\Gamma^z_{zz} = \frac{R'R''}{1+R'^2}
\]

(97)

Let us now examine what the general asymptotics presented in Section 6.1 become in the particular case of slow diffusions on \( \Sigma \). To simplify the arguments, we will restrict the discussion to diffusions that take place ‘along the \( Oz \) axis’. More precisely, we will suppose that the distribution function \( \Pi \) describing the diffusion depends only on \( z \) (and not on \( \theta \)) and that \( j_\theta = 0 \) everywhere on \( \Sigma \). These diffusions respect the axial symmetry of \( \Sigma \) and are surely the simplest diffusions possible on the considered surface.

The surface \( \Sigma \) enters the definition of only one out of the four small parameters considered in Section 6.1; this parameter has been denoted by \( \nu \) and fixes the scaling of the action of \( \nabla \) on the currents \( j \) and on the stress-energy tensor \( T \). To determine \( \nu \), let us evaluate the right-hand side of (84). Since the metric \( g \) is covariantly constant:

\[
\nabla_i \left( g^{ik} j_k \right) = g^{ik} \nabla_i j_k = g^{ik} \left( \frac{\partial j_k}{\partial x^i} - \Gamma^q_{ik} j_q \right).
\]

(98)

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Since we are only considering diffusion ‘along the $Oz$ axis’, $j_\theta = 0$; considering also that the metric $g$ is diagonal in the chosen coordinate system, one is led to:

$$g^{ik} \nabla_i j_k = g^{zz} \frac{\partial j_z}{\partial z} - j_z \left( g^{zz} \Gamma^z_{zz} + g^{\theta\theta} \Gamma^z_{\theta\theta} \right).$$  \hfill (99)

Expliciting the various contributions to the right-hand side of this equation, one finds:

$$\nabla_i (g^{ik} j_k) = \frac{1}{1 + R'^2} \left( \frac{\partial}{\partial z} + \frac{R'}{R} - \frac{R'R''}{1 + R'^2} \right) j_z.$$  \hfill (100)

Comparing (100) with (84), one sees that $\nu$ is fixed by the characteristic variation scales of both the current and the radius $R$. To make this point even clearer, let us choose the following, very particular form for $R$:

$$R(z) = R_0 + a \cos \left( \frac{z}{l} \right), \quad a, R_0 > 0,$$  \hfill (101)

with:

$$\frac{a}{R_0} = O(\alpha),$$  \hfill (102)

$$\frac{l}{R_0} = O(1),$$  \hfill (103)

$$\frac{\lambda}{R_0} = O(\alpha^2),$$  \hfill (104)

and:

$$\alpha \ll 1.$$  \hfill (105)

The radius of the cylinder is then practically constant, equal to $R_0$. With the simple choices of scalings (102) and (104), the small parameter $\alpha$ fixes both the amplitude of the variation of $R$ around $R_0$ and the (very small) ratio of the mean free-path $\lambda$ to the mean radius $R_0$. Finally, the radius $R$ is supposed to vary around $R_0$ on a characteristic length $l$ which is of the same order of magnitude as the mean radius $R_0$ itself. These scalings entering the definition of $R$ permit a very simple analysis and assessment of the main effects curvature can have on diffusion.

The scalings (102), (103) and (104) imply:

$$\frac{R'}{R} = \frac{1}{R_0} O(\alpha)$$  \hfill (106)
and
\[
\frac{R' R''}{1 + R'^2} = \frac{1}{R_0} O(\alpha). \tag{107}
\]

It follows from these relations that :
\[
\nu = \max(\epsilon, \alpha^3). \tag{108}
\]

The parameter \(\alpha\) enters the definition of \(R\) and is characteristic of the surface on which diffusion is studied. The parameter \(\epsilon\), on the other hand, fixes the spatial variation scale \(L\) of the density \(n\); by definition, \(\lambda/L = O(\epsilon)\).

Let \(L_c\) be defined by :
\[
L_c = \frac{\lambda}{\alpha^3}. \tag{109}
\]

\(L_c\) is the critical length around which curvature effects become important. Indeed, diffusions of characteristic (spatial) scale \(L \ll L_c\) ‘do not see’ the curvature of the surface; for these diffusions, \(\nu = \epsilon\) and \(\eta = \epsilon^2\), just as in Euclidean space. This scaling even suggests a diffusion equation of the form :
\[
\partial_t n = \chi \partial_{zz} n. \tag{110}
\]

On the other hand, diffusions of characteristic length \(L \gg L_c\) ‘are sensitive’ to curvature. For these diffusions, \(\nu = \alpha^3\) and, therefore, \(\eta = \alpha^3 \epsilon\). This suggests that the density \(n\) verifies an equation of the type :
\[
\partial_t n = \tilde{\chi} \partial_z n, \tag{111}
\]

the coefficient \(\tilde{\chi}\) depending generally on the point on the surface. Actually, curvature effects are important for all diffusions of characteristic length \(L\) not negligible before \(L_c\).

7 Conclusion

The problem of particle diffusion on an arbitrary, possibly curved 2D surface is a priori susceptible of various, very different approaches. Standard particle diffusion in 3D flat Euclidean-space is commonly described by stochastic processes; in particular, a well-known and practical model of free particle diffusion in 3D flat space is the Ornstein-Uhlenbeck process. This process is usually defined by a couple of stochastic equations, which fix the time-evolution of both the position and momentum of the diffusing particle. Alternately, the usual Ornstein-Uhlenbeck process can also be characterized by its associated Chapman-Kolmogorov equation, which fixes the
time-evolution of the probability distribution function defined on the phase-space of the diffusing particle. In the present article, we have proposed a generalization of this Ornstein-Uhlenbeck approach to the case of 2D free particle diffusion on an arbitrary surface.

After introducing the necessary geometrical tools (metric, connection,...), we have defined with precision the 4D phase-space of a particle diffusing on a 2D surface and the natural ‘volume’-measure to be used on this phase-space. The diffusion itself has been characterized by a generalization of the Chapman-Kolmogorov equation associated to the standard Ornstein-Uhlenbeck process. This new transport equation fixes the time-evolution of the probability distribution function of the diffusing particle in its 4D phase-space. Curvature effects are taken into account through the non-flat metric-tensor of the surface and through the (spatial) derivatives of this tensor, essentially represented by the connexion-coefficients aka. Christoffel symbols.

We have proved that the transport equation admits an equilibrium solution. We have also deduced from the transport equation the macroscopic balance equations which describe the conservation of the diffusing particle number and the net exchanges of energy/momentum between these particles and the fluid which surrounds them. These balance equations have been used to study qualitatively the so-called hydrodynamic limit of the diffusion process, which corresponds to ‘slow’ spatial and temporal variations of the spatial density $n$ of the diffusing particle. It has been shown that, at least in this hydrodynamic limit, diffusion on a curved surface can be very different from diffusion on flat space. In particular, the standard diffusion equation, which connects the first time-derivative of the particle density $n$ to the second space-derivatives of $n$, may have to be replaced by a non-standard ‘diffusion’ equation which, at leading order, involves only the first space-derivatives of $n$. The coefficient(s) of this new ‘diffusion’ equation generally depend on the position on the surface. Finally, these very general, abstract conclusions have been illustrated by the perhaps academic but very pedagogical example of diffusion on a cylinder of altitude-dependent radius.

The work presented in this article can be extended in various directions. First, a complete and general perturbative treatment of the hydrodynamic limit through a Chapman-Enskog expansion should be undertaken to gain more insights on what kind of generic modifications of the standard diffusion equation can be induced by the curvature of the surface on which the particle diffuses.

Second, one should also be able to construct a conditional entropy [14, 9] and a conditional entropy-current for the new model of 2D diffusion intro-
duced in this article. If that is indeed feasible, and how the curvature effects enter the expression of the conditional entropy, are two questions left open for further study.

Third, the problem of non-free particle diffusion on an arbitrary surface should also be examined. This could lead, for example, to generalizations of the famous Smoluchowski equation [15, 13, 1].

As mentioned in the Introduction, these generalizations may have significant consequences in biology, where processes often occur in complex media or curved interfaces, which, furthermore, are usually moving or fluctuating.

Another important question is therefore how a time-dependent geometry influences diffusion. This issue actually hides many theoretical problems of great interest. For instance: is there still a kind of \( H \)-theorem which characterizes the irreversibility of the process, even when the geometry of the surface on which the diffusion occurs varies with time? We hope to address this and other related questions in forthcoming publications.

References


